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Products of Involutory Matrices Over Rings

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ABSTRACT

Let $\alpha \in \mathrm{GL}_n A$ be a matrix over a commutative ring A with 1 such that $(\det \alpha)^2 = 1$. If α is cyclic, it can be written as a product of at most three involutions. When A satisfies the first Bass stable range condition, then α can be written as a product of at most five involutions. If in addition either $n \leqslant 3$ or $n = 4$ and $\det \alpha = -1$, then α can be written as a product of at most four involutions. When A is a Dedekind ring of arithmetic type, the number of involutions needed to express α is uniformly bounded for any $n \geqslant 3$. When $A = \mathbf{C}[x]$ the number of involutions is unbounded for any $n \geq 2$.

An *involution* in a group G is an element α of order 2. For any $\beta \in G$, the conjugate $\beta\alpha\beta^{-1}$ is also an involution, so it follows that the set of all

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products of involutions is a normal subgroup H of G. Thus if G is simple and H is nontrivial, then $H = G$.

An element of a group G is called k-reflectional if it can be written as a product of at most k involutions, that is, k or fewer involutions. We say that G is k-reflectional if every element of G which is a product of involutions can be written as a product of at most k involutions.

Every element of S_n, the group of permutations on n letters, is 2-reflectional (see [9]). On the other hand, this is not true in general for the alternating group A_n. For $n \geqslant 5$, A_n is 3-reflectional, and in [5], Berggren showed that only for $n = 5, 6, 10$, or 14 is A_n 2-reflectional.

For an associative ring A with 1, denote by $\mathrm{GL}_n A$ the group of invertible n by n matrices with entries in A. For commutative A define $\mathrm{IL}_n A$ to be the subgroup of $\mathrm{GL}_n A$ containing all matrices α such that $(\det \alpha)^2 = 1$. This subgroup contains the normal subgroup H of $\mathrm{GL}_n A$ formed by all products of involutions. When A is a field, it was shown in [10] that every matrix of determinant 1 or -1 is in H and thus $H = \pm \mathrm{SL}_n A = \mathrm{IL}_n A$.

In [18], Waterhouse showed that any elementary matrix is 2-reflectional, and his proof is valid over any associative ring with 1. Thus the group $\mathrm{E}_n A$ generated by elementary matrices is contained in H. Gustafson showed in [9] that when $\mathrm{E}_n A = \mathrm{SL}_n A$ and A is commutative, $H = \mathrm{IL}_n A$.

We can show that $H = \mathrm{IL}_n A$ for any commutative ring A with $\mathrm{SL}_n A \subset H$ by the following argument. Let $\alpha \in \mathrm{IL}_n A$, and δ be a diagonal matrix which differs from the identity matrix in one diagonal entry, which is $\det \alpha$. Then $\alpha = (\alpha\delta)\delta$, δ is an involution, and $\alpha\delta \in \mathrm{SL}_n A$. We will show that $H = \mathrm{IL}_n A$ when A is commutative and satisfies the first Bass stable range condition.

In [10], it was shown that over a field F, a matrix in $\mathrm{GL}_n F$ is 4-reflectional if and only if it has determinant 1 or -1. Furthermore, four is the smallest such number. In [16], Sourour gave a short proof of this for the special case where F has at least $n + 2$ elements, and Laffey [14] obtained a similar result without the restriction on the size of F. This implies that every matrix in $\mathrm{GL}_n F$ which is the product of involutions is 4-reflectional. Knüppel and Nielsen showed in [13] that $\mathrm{SL}_n F$ is 4-reflectional provided $n \neq 2$, and established under what conditions $\mathrm{SL}_n F$ would be 3-reflectional. This was an extension of Ballantine's results in [4] for conditions under which $\mathrm{GL}_n F$ is 3-reflectional. In [1] and [2], products of involutions in $\mathrm{GL}_2 F$ and the projective group $\mathrm{PSL}_2 F$ with F a field of characteristic not 2 are studied. In [1], it is shown that a matrix in $\mathrm{GL}_n F$ of determinant 1 or -1 has the form $\alpha\beta\alpha\beta$, where α and β are involutions. This holds true in $\mathrm{PSL}_2 F$ provided $|F|$ is odd and greater than 3. In [2], it is shown that $\mathrm{PSL}_2 F$ is not 2-reflectional if -1 is not a square.

It is only natural to ask which matrices can be written as products of smaller numbers of involutions. Over a field with characteristic not 2, Wonnenburger showed in [19] that a matrix α is 2-reflectional if and only if α is similar to α^{-1}. Djokivić later proved this in [7] for arbitrary (commutative) fields. This result was also obtained by Ballantine [3] and by Hoffman and Paige [11]. While it is easy to show in any group that α is similar to α^{-1} if α is 2-reflectional, Ellers showed in [8] that the converse cannot be proven for matrices over noncommutative fields. For example, over the quaternions, the matrix

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

is similar to its inverse but is not 2-reflectional. Such a matrix exists over the quaternions for any integer $n \leqslant 2$. Wu [20] includes a discussion of related results over complex Hilbert spaces and poses the question of whether a bounded linear operator which is similar to its inverse is the product of two involutions.

There are no results which completely describe the matrices over a field which are 3-reflectional. However, there are large classes of matrices which are 3-reflectional. In [4], Ballantine proved that every matrix of determinant 1 or -1 having no more than two nontrivial invariant factors is 3-reflectional over F. Other examples include direct sums of cyclic matrices [4, 15].

Over more general rings, there are fewer results. The most significant of these appears in [12], where Knüppel uses a result of [17] to show that over a commutative stable range 1 ring A, $\pm \mathrm{SL}_n A$ is 5-reflectional. However, $\pm \mathrm{SL}_n A$ does not include all products of involutions, as A may include many elements of order 2.

We will demonstrate in this paper that for A a commutative ring with 1 satisfying the first Bass stable range condition, every matrix in $\mathrm{GL}_n A$ that is the product of involutions is 5-reflectional. We will also show that over any commutative ring A with 1, any cyclic matrix in $\mathrm{IL}_n A$ is 3-reflectional.

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + \alpha_0 \in A[x]$ be a monic polynomial with coefficients in A. Then its *companion matrix* $C(a_0, \ldots, a_{n-1})$ is the n by n matrix over A with ones along the diagonal just below the main diagonal, the elements $-a_0, \ldots, -a_{n-1}$ as the entries of the last column, and zeros elsewhere. Sometimes the transpose is called the companion matrix, but this makes no difference, as the companion matrix is similar to its transpose.

A matrix $\alpha \in M_n A$ is called *cyclic* if there exists a column $v \in A^n$ such that the vectors $v, \alpha v, \ldots, \alpha^{n-1}v$ form a basis for A^n. In this case, $\rho^{-1}\alpha\rho$ is a companion matrix, where $\rho = (v, \alpha v, \ldots, \alpha^{n-1}v) \in \mathrm{GL}_n A$. Conversely, whenever α is similar to a companion matrix, such a column v exists. Thus a matrix is cyclic if and only if it is similar to a companion matrix.

The polynomial $f(x)$ defined above is called *reciprocal* if there exists a unit $\lambda \in A$ such that $x^n f(x^{-1}) = \lambda f(x)$. When $f(x)$ is monic, this clearly requires that $\lambda^2 = 1$.

We will use the following elementary result without reference: if a matrix α is k-reflectional, and β is similar to α, then β is k-reflectional. This allows us to replace a cyclic matrix with the corresponding companion matrix in the proofs of Proposition 1 and Lemma 2.

PROPOSITION 1 (cf. [12]). *Let A be a commutative ring with* 1, *and let $\varphi \in \mathrm{GL}_n A$ be a cyclic matrix with reciprocal characteristic polynomial $q(x)$. Then φ is 2-reflectional.*

Proof. The matrix φ is similar to the companion matrix

$$\begin{pmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix},$$

which can be written as the product $\beta\pi$, with

$$\beta = \begin{pmatrix} -a_0 & & & 0 \\ -a_1 & & & 1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ -a_{n-1} & 1 & & 0 \end{pmatrix}$$

and permutation matrix

$$\pi = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

Obviously, π is an involution, and since $q(x)$ is reciprocal, $a_0 a_i = a_{n-i}$ for every i. Thus β is also an involution. ∎

In the case where A is a field, it was shown [4] that every cyclic matrix with determinant 1 or -1 is 3-reflectional. We extend this result to an arbitrary commutative ring A with 1 by showing that a cyclic matrix in $\mathrm{IL}_n A$ can be multiplied by an involution to obtain a matrix satisfying the hypotheses of Proposition 1.

LEMMA 2 (cf. [12]). *Let A be a commutative ring with* 1, *and let* $\gamma \in \mathrm{GL}_n A$ *be a cyclic matrix. Let* $q(x) = \Sigma c_i x^i$ *a monic polynomial such that* $-c_0 = (-1)^n \det \gamma$. *Then there is an involution* ρ *such that* $\varphi = \rho\gamma$ *is cyclic with characteristic polynomial* $q(x)$.

Proof. The matrix γ is similar to the companion matrix

$$\begin{pmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix},$$

so we can call this companion matrix γ.

Let

$$\rho = \begin{pmatrix} -1 & & & 0 \\ d_1 & 1 & & \\ \vdots & & \ddots & \\ d_{n-1} & & & 1 \end{pmatrix},$$

where $d_i = (c_i - a_i)a_0^{-1}$, with $1 \leqslant i \leqslant n$. Then we have $\rho\gamma = \varphi$, where φ is the companion matrix

$$\varphi = \begin{pmatrix} 1 & & & -c_0 \\ 1 & & & -c_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -c_{n-1} \end{pmatrix}.$$

Thus φ is cyclic with characteristic polynomial $q(x)$, and ρ is an involution. ∎

The only condition on the polynomial $q(x)$ in Lemma 2 is on its constant term. If the matrix $\gamma \in \mathrm{IL}_n A$, then it has the correct determinant; hence we can choose $q(x)$ to be reciprocal, and apply Lemma 1 to obtain the following generalization of the corresponding result over fields.

PROPOSITION 3. *Let A be a commutative ring with* 1, *and let* $\gamma \in \mathrm{IL}_n A$ *be a cyclic matrix. Then* γ *is* 3-*reflectional.*

Proof. Since $\gamma \in \mathrm{IL}_n A$, we can apply Lemma 2 with a reciprocal polynomial $q(x)$. The resulting matrix φ is 2-reflectional by Proposition 1, and since $\gamma = \rho\varphi$, γ is 3-reflectional. ∎

Now the full power of these propositions can be used when we can factor a matrix in $\mathrm{IL}_n A$ into a product of involutions and cyclic matrices in $\mathrm{IL}_n A$. Over rings satisfying the first Bass stable range condition, Vaserstein and Wheland obtained such a result in [17].

THEOREM 4 [17, Theorem 2]. *Let $\alpha \in \mathrm{GL}_n A$, where A is an associative ring with* 1 *satisfying the first Bass stable range condition. Then we can write $\alpha = \beta\gamma$, where β and γ are cyclic matrices and β is similar to any prescribed invertible companion matrix.*

This was proved using the following two lemmas.

LEMMA 5 [17, Theorem 1]. *Let A be an associative ring with* 1 *satisfying the first Bass stable range condition. Then every matrix in $\mathrm{GL}_n A$ is similar to the product of an upper triangular matrix and a lower triangular matrix.*

LEMMA 6 [17, Proposition 9]. *Let A be an associative ring with* 1, *ρ an invertible companion matrix in $\mathrm{GL}_n A$, μ an upper triangular matrix in $\mathrm{GL}_n A$, and γ a lower triangular matrix in $\mathrm{GL}_n A$. Assume that all diagonal entries of γ and μ are invertible. Then there is a matrix β similar to ρ and a cyclic matrix γ such that $\mu\lambda = \beta\gamma$.*

This allows us to apply our previous results to the case where A has stable range one, and thus to improve on the results in [12] with our main theorem.

THEOREM 7. *Let A be a commutative ring with* 1 *satisfying the first Bass stable range condition, and let $\alpha \in \mathrm{IL}_n A$. Then α is* 5*-reflectional.*

Proof. As in Theorem 4, write $\alpha = \beta\gamma$, where β is a cyclic matrix with reciprocal characteristic polynomial and γ is a cyclic matrix in $\mathrm{IL}_n A$. Then β is 2-reflectional by Proposition 1, and γ is 3-reflectional by Proposition 3. Hence α is 5-reflectional. ∎

THEOREM 8. *Let A be a commutative ring with* 1 *satisfying the first Bass stable range condition, and let $\alpha \in \mathrm{IL}_n A$. Assume that $n \leqslant 3$ or $n = 4$ and* $\det \alpha = -1$. *Then α is* 4*-reflectional.*

Proof. For $n = 1$, it's obvious. For $n = 2$, multiply α by the involution $\operatorname{diag}(-\det \alpha, 1)$ to obtain β with $\det \beta = -1$. Then we will show that β is 3-reflectional. By the stable range condition, β is similar to a matrix whose upper left corner entry is invertible. Then by multiplying by the involution π, we can make the upper right corner entry invertible. This matrix is similar to a matrix with 1 and 0 in the second column. Since the determinant is now 1, the lower left corner must be -1. Multiplying this by the involution π mentioned above, we get an involution. Thus β is 3-reflectional and α is 4-reflectional.

For $n = 3$, we can replace α by a matrix whose upper left corner entry is invertible. Multiplying this by the involution obtained from π by replacing the center entry with the negative of the determinant of α, we can make the lower left entry invertible and the determinant 1. This matrix is similar to one whose upper right corner is invertible, which is similar to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ v & * & * \\ * & * & * \end{pmatrix},$$

where v can be made a unit by the stable range condition. Multiplying by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

we obtain a matrix which is similar to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ * & w & * \end{pmatrix}.$$

Since the determinant is 1, $w = -1$. Multiplying by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & -1 \end{pmatrix},$$

which is an involution, and thus α is 4-reflectional.

For $n = 4$, let $\det \alpha = -1$. We can write α as the matrix

$$\begin{pmatrix} U & * \\ * & * \end{pmatrix},$$

where U is a 2 by 2 matrix which can be chosen by the stable range condition to be invertible. Multiplying by

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

we obtain a matrix which is similar to

$$\begin{pmatrix} 0 & I \\ * & * \end{pmatrix}.$$

We can use the stable range condition to find an appropriate k so that this matrix is similar to

$$\begin{pmatrix} 0 & 0 & 1 & -k \\ 0 & 0 & 0 & -1 \\ * & u & * & * \\ * & * & * & * \end{pmatrix},$$

where u is invertible. Multiplying by

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} 1 & -k & 0 & 0 \\ 0 & -1 & 0 & 0 \\ * & * & * & u \\ * & * & * & * \end{pmatrix},$$

which is similar to

$$\begin{pmatrix} 1 & -k & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ * & * & -1 & * \end{pmatrix}.$$

This matrix can be written as the product of the involutions

$$\begin{pmatrix} 1 & -k & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & * & * & -1 \end{pmatrix}.$$

Thus α is 4-reflectional. ■

Let A be a Dedekind ring of arithmetic type. From [6, Theorem 20], we obtain the following result:

LEMMA 9. *Let A be a Dedekind ring of arithmetic type, $n \geqslant 3$, $\alpha \in \mathrm{SL}_n A$. Then α is a product of a bounded number of (upper or lower) u· it triangular matrices.*

LEMMA 10. *Let A be an associative ring with* 1, *and α a product of m unit upper and lower triangular matrices in $\mathrm{GL}_n A$. Then α is a product of at most* $\max\{5, 2.5m\}$ *involutions.*

Proof. If α is a product of an odd number m of matrices, $m \geqslant 3$, we can use similarity to replace α by a matrix which is a product of $m - 1$ matrices. Therefore, if $m \neq 1$, it suffices to consider the case where m is even. The product of an unit upper and an unit lower triangular matrix can be written as a product of five involutions by Propositions 1 and 3 and Lemma 6, so we are left with the case $m = 1$. Since the lower triangular case is similar to the upper triangular case, it suffices to consider α a unit upper triangular matrix. If we multiply α by two cyclic permutation matrices, we obtain a cyclic matrix, which is 3-reflectional by Proposition 3. Thus α is 5-reflectional. ■

THEOREM 11. *Let A be a Dedekind ring of arithmetic type. Then there exists a number k such that $\mathrm{GL}_n A$ is k-reflectional for any $n \geqslant 3$.*

Proof. Let $\alpha \in \mathrm{GL}_n A$ be a product of involutions. Let β be the product of α and $\operatorname{diag}(\det \alpha, 1, \ldots, 1)$, so $\beta \in \mathrm{SL}_n A$. By Lemma 9, we can write β as a product of m unit triangular matrices. By Lemma 10, β is k-reflectional, where $k \leqslant \max\{5, 2.5m\}$. Thus α is $(k + 1)$-reflectional. ■

REMARK. Over a ring of algebraic integers, for large n, 15 involutions is enough, since by [6, Theorem 20] for sufficiently large n any matrix in $\mathrm{SL}_n A$

can be written as a product of six triangular matrices. The bound 15 seems to be far from the best possible.

If A is a commutative ring with 1, it is only natural to ask whether an absolute constant k exists such that $\mathrm{GL}_n A$ is k-reflectional. We show that the answer in general is negative. Moreover, we show that there is no bound on the number of involutions even if we restrict ourselves to the Euclidean polynomial ring $\mathbf{C}[x]$ with complex coefficients and fix an arbitrary $n \geqslant 2$.

LEMMA 12. *Let A be a commutative ring such that 2 is invertible in A and all projective A-modules are free. If $\mathrm{GL}_n A$ is k-reflectional for some $k < \infty$, then $c(\mathrm{GL}_n A) \leqslant k$.*

Proof. Suppose α is in the commutator subgroup of $\mathrm{GL}_n A$. We want to prove that α is a product of k commutators. By our assumption, α is k-reflectional. So $\alpha = \pi_1 \pi_2 \cdots \pi_k$, where $\pi_i^2 = I$. We can arrange the π_i by type $d_i = \mathrm{rank}(\pi_i + 1)$ so that all pairs of the same type are together, and the left-over involutions of odd types are grouped together (the left-over involutions of even types can be put at the end of the product). Any involution of type d_i is similar to the diagonal matrix having $n - d_i$ ones. Since involutions of the same type are similar to one another, any product of two of them is a commutator. If d_i is even, the involution of type d_i is a commutator. Since $\alpha \in \mathrm{SL}_n A$, there are an even number of involutions of distinct odd types remaining. If we take some pair π_i and π_j of these of types d_i, $j > i$, then we can write π_j as a product of two involutions, one of type d_i, and the other of type d_{j-i}. So we now have a pair of involutions of type d_i, and another involution of even type. Thus this is the product of two commutators. This means that α is a product of at most k commutators. ∎

THEOREM 13. *Let $n \geqslant 2$. Then there is no k such that $\mathrm{GL}_n(\mathbf{C}[x])$ is k-reflectional.*

Proof. Suppose the conclusion is false, i.e., there exists some k such that $\mathrm{GL}_n(\mathbf{C}[x])$ is k-reflectional. By [6, Theorem 1], we know that there is a matrix α in the group $\mathrm{SL}_n(\mathbf{C}[x])$ which cannot be written as a product of k commutators. Thus by Lemma 12 we obtain a contradiction. ∎

REMARK. The field $\mathbf{C}$ in Theorem 13 can be replaced by any field of infinite transcendence degree over its prime field of characteristic not 2 (see [6, Theorem 1]).

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